

On Feynman rules for Mellin amplitudes in AdS/CFT

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Abstract

The computation of CFT correlation functions via Witten diagrams in AdS space can be simplified via the Mellin transform. Recently a set of Feynman rules for tree-level Mellin space amplitudes has been proposed for scalar theories. In this note we derive these rules by explicitly evaluating all of the relevant Witten diagram integrals for the scalar ϕ^n theory. We also check that the rules reduce to the usual Feynman rules in the flat space limit.

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I. INTRODUCTION

AdS/CFT is a powerful tool [1–3] which among other things allows us to compute CFT correlators at strong coupling via Witten diagrams in AdS space. In practice these computations are still quite challenging in position space and generally require a lot of work, see [4–20]. Recently it has been argued that taking the Mellin transform of correlation functions drastically simplifies the computations and the resulting expressions have nice mathematical structure, see e.g. [21–28]. The correlation functions of primary scalar operators for a CFT can be written in Mellin space as

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\dots\mathcal{O}(x_n) \rangle \sim \int d\delta_{ij} M(\delta_{ij}) \prod_{1 \leq i < j \leq n} \Gamma(\delta_{ij}) (x_{ij}^2)^{-\delta_{ij}}, \quad (1.1)$$

where $M(\delta_{ij})$ is called the Mellin amplitude and parameters δ_{ij} can be parametrized as $\delta_{ij} = k_i \cdot k_j$. Mellin amplitudes have many similarities to scattering amplitudes in flat space, in particular the large AdS radius limit of the Mellin amplitude was argued to be equivalent to the scattering amplitudes in flat space [23, 25], such that k_i plays the role of momentum in flat space,¹ suggesting that Mellin amplitudes could be used to provide a holographic definition of the S-matrix, see [29–38] for related discussion.

More recently in [24] and [25], the authors studied various aspects of the Mellin representation of AdS correlators. In particular, a set of Feynman rules, for computing Mellin amplitudes for any theory of scalar field at tree-level, was proposed and checked for a few non-trivial correlators in ϕ^3 and ϕ^4 theories in [24] as well as recursively via a factorization formula for ϕ^3 theory in [25].

In this note we will consider a scalar field with ϕ^n interaction at tree level and offer a direct proof of the Feynman rules for Mellin amplitudes by evaluating all the Witten diagram integrals explicitly. We hope that our results will be useful for better understanding of the structure of Mellin amplitudes and for the future development of similar rules for fields with spin and for loop amplitudes. We have also checked that the Mellin space Feynman rules reduce to the usual Feynman rules in the flat space limit.

The paper is organized as follows. In section 2 we review the conjectured Feynman rules for Mellin amplitudes. In section 3 we use particular forms of the bulk-to-boundary and bulk-to-bulk propagators to compute the Witten diagram with the maximal off-shell vertex for a ϕ^n theory and show that they lead to the conjectured Feynman rules for Mellin amplitudes. Then we demonstrate

¹ The quantities k_i ’s are also called “Mellin momentum” or “fictitious momentum” but for the sake of brevity we will just refer to them as “momentum” in the rest of the paper and we stress that this interpretation is precise only when we look at the flat space limit of the Mellin amplitude and not in general.

that we get the same Feynman rules for such a vertex embedded in a very general Witten diagram of the ϕ^n theory. In section 4 we show that these Feynman rules reduce to the usual Feynman rules in the flat space limit.

Note added: While this paper was in preparation, the paper [40] appeared which checks the formula for the off-shell n -pt vertex of the ϕ^n theory via recursion relations.

II. FEYNMAN RULES FOR MELLIN AMPLITUDES

Let us first review the Feynman rules for Mellin amplitudes corresponding to any tree level Witten diagram in AdS_{d+1} for a ϕ^n scalar theory, as proposed in [24]. To compute the Mellin amplitude one has to put together propagators and vertices following a few simple steps:

- Assign a “momentum” k_i to every line such that the external lines of the Witten diagram have $-k_i^2 = \Delta_i$ and at each vertex we have conservation $\sum_i k_i = 0$,² where Δ_i is the conformal dimension of the corresponding field.

- Assign an integer m_i to each internal line with the propagator

$$\mathcal{P}_i = \frac{-1}{2m_i! \Gamma(1 + \Delta_{m_i} + m_i - h)} \frac{1}{k_i^2 + (\Delta_{m_i} + 2m_i)}, \quad (2.1)$$

where $h = d/2$.

- The factor for a vertex connecting lines with dimension Δ_i and integers m_i , (see Fig. 1) is given by

$$V_{[m_1, \dots, m_n]}^{\Delta_1, \dots, \Delta_n} = g^{(n)} \Gamma\left(\frac{\sum_{i=1}^n \Delta_i - 2h}{2}\right) \left(\prod_{i=1}^n (1 - h + \Delta_i)_{m_i}\right) F_A^{(n)}\left(\frac{\sum_{i=1}^n \Delta_i - 2h}{2}, \{-m_1, \dots, -m_n\}, \{1 + \Delta_1 - h, \dots, 1 + \Delta_n - h\}; 1, \dots, 1\right) \quad (2.2)$$

where $g^{(n)}$ is the coupling in the $g^{(n)}\phi^n$ theory, $(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}$ is the Pochhammer symbol and $F_A^{(n)}$ is the Lauricella function of n variables

$$F_A^{(n)}(y, \{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}; x_1, \dots, x_n) = \sum_{l_i=0}^{\infty} \left((y)^{\sum_{i=1}^n l_i} \prod_{i=1}^n \frac{(a_i)_{l_i}}{(b_i)_{l_i}} \frac{x_i^{l_i}}{l_i!} \right). \quad (2.3)$$

- Finally, sum over all positive integers m_i to obtain the Mellin amplitude.

We note here that the vertex given above is the most general type of vertex (or the maximal off-shell vertex) when all legs are off-shell³, but the theory would also have vertices with less number

² The vector k has such properties because it solves the constraints of δ_{ij} , namely $\delta_{ij} = \delta_{ji}$ and $\sum_{j \neq i} \delta_{ij} = -\Delta_i^2$.

³ The legs connecting to the AdS boundary directly are referred to as the on-shell legs, while those that do not connect to the boundary are the off-shell legs.

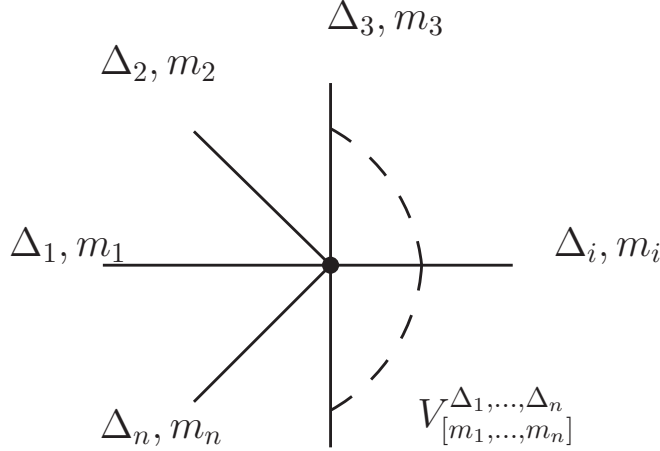


FIG. 1: A general vertex for ϕ^n theory.

of off-shell legs. The vertex factor in such cases can be simply obtained from the general case by taking some of the m_i 's to zero, corresponding to the legs going on-shell. Also, note that the Lauricella function of m variables can be written in a series form as in (2.3) which is convergent for $\sum_i |x_i| < 1$. For the vertex above, all n variables x_i take a particular value 1, which is the Lauricella function evaluated at that particular point, which is well-defined via analytic continuation.

III. PROOF OF FEYNMAN RULES

A. Maximal off-shell vertex

In this section we consider a Witten diagram for the scalar ϕ^n theory in AdS_{d+1} which has a vertex with the maximal number of off-shell legs (see Fig. 2) and prove the Feynman rules for this case which we described in the previous section.

Let X^I be the coordinates of the Euclidean AdS_{d+1} space, embedded in a $d + 2$ dimensional Minkowski space such that $X^2 = -R^2$, where R is the AdS radius and the point on the boundary P^A defined on the light-cone such that $P^2 = 0$. The bulk-to-boundary propagator between a point P on the boundary and X in the bulk for a scalar field of dimension Δ is given by⁴

$$E(P, X) = \frac{1}{2\pi^h \Gamma(1 + \Delta - h)} \int_0^{+\infty} \frac{dt}{t} t^\Delta e^{2tP \cdot X}. \quad (3.1)$$

⁴ We will drop the normalization factor $\frac{1}{2\pi^h \Gamma(1 + \Delta - h)}$ from (3.1) for subsequent calculations, since it is not relevant. Moreover this factor goes into the overall normalization factor in the definition of Mellin amplitudes and according to (1.1) we have ignored it in this note and the inclusion of this factor would allow us to write (1.1) as an equality.

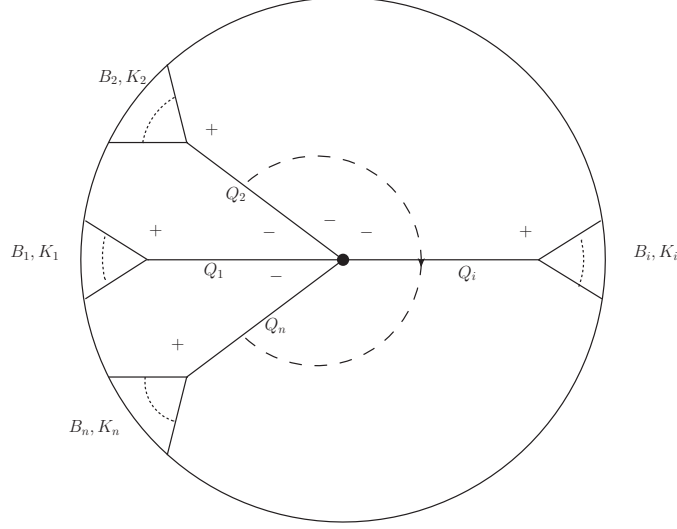


FIG. 2: The Witten diagram for the scalar ϕ^n theory with a vertex having n off-shell legs and n vertices having 1 off-shell leg and $(n-1)$ on-shell legs.

The bulk-to-bulk propagator between the points X_1 and X_2 can be written as an integral over a point Q on the boundary of the AdS, the integrand being the product of two bulk-to-boundary propagators of states with unphysical dimension $h \pm c$

$$G_{BB}(X_1, X_2) = \int_{-i\infty}^{+i\infty} \frac{dc}{2\pi i} f_{\Delta}(c) \int_{\partial AdS} dQ \int \frac{ds}{s} \frac{d\bar{s}}{\bar{s}} s^{h+c} \bar{s}^{h-c} e^{2sQ \cdot X_1 + 2\bar{s}Q \cdot X_2} \quad (3.2)$$

where

$$f_{\Delta}(c) \equiv \frac{1}{2\pi^{2h}[(\Delta - h)^2 - c^2]} \frac{1}{\Gamma(c)\Gamma(-c)}. \quad (3.3)$$

To simplify the notations, let us call each set of external $(n-1)$ legs in Fig. 2 as a block, and denote it as B_i with $i = 1, \dots, n$. Equations (3.2) and (3.1) give us the building blocks of any arbitrary Witten diagram in the ϕ^n theory. Fig. 2 can be constructed using two types of n -point correlation functions, built out of (3.1) and (3.2), which are given as

$$A_n(B_i, Q_{i,+}) = g^{(n)} \int_0^{+\infty} \prod_{i=1}^{n-1} \frac{dt_i}{t_i} t_i^{\Delta_i} \frac{ds_i}{s_i} s_i^{h+c_i} \int_{AdS} dX_i e^{2(B_i + s_i Q_i) \cdot X_i}, \quad (3.4)$$

$$A_n(Q_{1,-}, \dots, Q_{n,-}) = g^{(n)} \int_0^{+\infty} \prod_{i=1}^n \frac{d\bar{s}_i}{\bar{s}_i} \bar{s}_i^{h-c_i} \int_{AdS} dX_i e^{2(\sum_{k=1}^n \bar{s}_k Q_k) \cdot X_i},$$

where the blocks of $(n-1)$ legs, which we call B_i , are typically given as,

$$B_i = \sum_{k=(i-1)(n-1)+1}^{i(n-1)} t_k P_k. \quad (3.5)$$

We note that in general B_i can also contain fewer legs, (in which case the Witten diagram gives a vertex with fewer off-shell legs) so the limits of the summation in (3.5) would change according to the diagram under consideration. Moreover in Fig. 2 the label K_i indicates

$$K_i \equiv k_{a_i} + \dots + k_{a_i+n-1} \quad (3.6)$$

the sum of the momenta of all the fields in the block B_i where k_{a_i} is the momenta of each field.

Now, let us write the expression for the Witten diagram in Fig. 2 using the n -point correlation functions (3.4) as the building blocks and we get

$$\begin{aligned} A &= \int_{-i\infty}^{+i\infty} \prod_{i=1}^n \frac{dc_i}{(2\pi i)^n} f_{\Delta_i}(c_i) \int_{\partial AdS} \prod_{i=1}^n dQ_i \left(A_n(B_i, Q_{i,+}) \dots A_n(B_n, Q_{n,+}) A_n(Q_{1,-}, \dots, Q_{n,-}) \right) \\ &= (g^{(n)})^{n+1} \int_{-i\infty}^{+i\infty} \prod_{i=1}^n \frac{dc_i}{(2\pi i)^n} f_{\Delta_i}(c_i) \int \prod_{i=1}^{n(n-1)} \frac{dt_i}{t_i} t_i^{\Delta_i} \int \prod_{i=1}^n \frac{ds_i}{s_i} \frac{d\bar{s}_i}{\bar{s}_i} s_i^{h+c_i} \bar{s}_i^{h-c_i} \\ &\quad \times \int_{\partial AdS} \prod_{i=1}^n dQ_i \int_{AdS} \prod_{i=1}^{n+1} dX_i \exp \left(2 \sum_{i=1}^n X_i \cdot (B_i + s_i Q_i) + 2X_{n+1} \cdot \left(\sum_{i=1}^n \bar{s}_i Q_i \right) \right), \end{aligned} \quad (3.7)$$

where A is the $n(n-1)$ point correlation function.

B. Evaluating the integrals

Integrals over X_i

The integrations over the bulk points X_i can be done by applying (A1) from the Appendix, which gives us the result,

$$\begin{aligned} A &= (g^{(n)})^{n+1} (\pi^h)^{n+1} \int_{-i\infty}^{+i\infty} \prod_{i=1}^n \frac{dc_i}{(2\pi i)^n} f_{\Delta_i}(c_i) \int \prod_{i=1}^{n(n-1)} \frac{dt_i}{t_i} t_i^{\Delta_i} \int \prod_{i=1}^n \frac{ds_i}{s_i} \frac{d\bar{s}_i}{\bar{s}_i} s_i^{h+c_i} \bar{s}_i^{h-c_i} \\ &\quad \int_{\partial AdS} \prod_{i=1}^n dQ_i \left(\prod_{i=1}^n \Gamma \left(\frac{\Delta_{B_i} + (h + c_i) - 2h}{2} \right) \right) \Gamma \left(\frac{\sum_{i=1}^n (h - c_i) - 2h}{2} \right) e^{E_Q}, \end{aligned} \quad (3.8)$$

where $\Delta_{B_i} = \sum_{j \in B_i} \Delta_j$ and Δ_j is the conformal dimension of the field j and the exponent in the above integrand is given by,

$$E_Q = \sum_{i=1}^n (B_i + s_i Q_i)^2 + \left(\sum_{i=1}^n \bar{s}_i Q_i \right)^2. \quad (3.9)$$

Integrals over Q

To perform the Q_i integrals we first expand (3.9) and rewrite it as,

$$E_Q = \sum_{i=1}^n (B_i^2 + 2s_i Q_i \cdot B_i) + 2 \sum_{1 \leq i < j \leq n} \bar{s}_i \bar{s}_j Q_i \cdot Q_j. \quad (3.10)$$

Now we will integrate out the Q_i 's successively.

First we will do the Q_1 integral using (A2) and using the on-shell condition, $Q_i^2 = 0$ to simplify the result, we finally get,⁵

$$E_{Q_1} = \sum_{i=1}^n B_i^2 + (s_1 B_1)^2 + 2 \left((1 + \bar{s}_1^2) \left(\sum_{1 < i < j}^n \bar{s}_i \bar{s}_j Q_i \cdot Q_j \right) + \sum_{i \neq 1}^n (s_i B_i + \bar{s}_i (\bar{s}_1 s_1 B_1)) \cdot Q_i \right), \quad (3.11)$$

where the notation E_{Q_k} is used to denote the exponent obtained as a result of doing the set of successive integrals from Q_1 to Q_k , i.e.

$$\int_{\partial AdS} \prod_{i=1}^k dQ_i e^{E_Q} = e^{E_{Q_k}}.$$

Next, we perform the Q_2 integral in a similar way and we find that E_{Q_2} can be written as

$$E_{Q_2} = \sum_{i=1}^n B_i^2 + (s_1 B_1)^2 + (s_2 B_2 + \bar{s}_2 (\bar{s}_1 s_1 B_1))^2 + 2 \left((1 + \bar{s}_1^2)(1 + \bar{s}_2^2(1 + \bar{s}_1^2)) \left(\sum_{2 < i < j}^n \bar{s}_i \bar{s}_j Q_i \cdot Q_j \right) + 2 \left(\sum_{i \neq \{1,2\}}^n \left(s_i B_i + \bar{s}_i ((\bar{s}_1 s_1 B_1) + (1 + \bar{s}_1^2) \bar{s}_2 (s_2 B_2 + \bar{s}_2 (\bar{s}_1 s_1 B_1))) \right) \cdot Q_i \right) \right). \quad (3.12)$$

We continue integrating out the Q_i 's successively as in the last few steps and integrating the p th step the result is of the form,

$$E_{Q_p} = \sum_{i=1}^n B_i^2 + \sum_{i=1}^p (\bar{s}_i Y_{i-1} + s_i B_i)^2 + 2 \left(\left(\prod_{m=1}^p g_m \right) \left(\sum_{p < i < j}^n \bar{s}_i \bar{s}_j Q_i \cdot Q_j \right) + 2 \left(\sum_{i \neq \{1,2,\dots,p\}}^n (s_i B_i + \bar{s}_i Y_p) \cdot Q_i \right) \right), \quad (3.13)$$

where we have defined the functions Y_i and g_i as,

$$Y_l = \sum_{i=1}^l \frac{(\prod_{k=1}^l g_k)}{g_i} s_i \bar{s}_i B_i \quad \text{and} \quad (3.14)$$

$$g_l = (1 + \bar{s}_l^2 \prod_{k=0}^{l-1} g_k) \quad \text{with } g_0 = 1.$$

After integrating out all the Q_i 's using (3.13) we finally get the exponent of the integrand in (3.8) as,

$$E_{Q_n} = \sum_{i=1}^n B_i^2 + \sum_{l=1}^n (\bar{s}_l Y_{l-1} + s_l B_l)^2. \quad (3.15)$$

⁵ At each step of doing the Q integral we will get a $2\pi^h$ factor which we would drop in the following steps to help us reduce clutter.

Integrals over t_i

Let us first expand the term in the parentheses in (3.15), and with the help of (3.14) we get

$$E_{Q_n} = \sum_{i=1}^n (1 + s_i^2 F_i) B_i^2 + \sum_{1 \leq i < j \leq n} \frac{2(s_i \bar{s}_i s_j \bar{s}_j)(B_i \cdot B_j)}{g_i g_j} \left(\prod_{l=1}^n g_l \right). \quad (3.16)$$

where

$$F_i = 1 + \frac{\bar{s}_i^2}{g_i^2} \left(\sum_{l=i+1}^n \bar{s}_l^2 \left(\prod_{k=1}^{l-1} g_k^2 \right) \right). \quad (3.17)$$

Next, we will apply Symanzik star formula (A3) to our integral (3.8) in order to obtain the Mellin amplitude $M(\delta_{ij})$.

Let us recall that B_i 's are given as $\sum t_l P_l$, so the exponent of the integrand would only have terms quadratic in t coming from expanding the B_i^2 and $B_i \cdot B_j$ terms in (3.16)⁶. Using (A3), we can see that the full result of the Witten diagram integral gives,

$$A = (g^{(n)})^{n+1} \frac{(\pi^h)^{n+2}}{2(2\pi i)^{\frac{1}{2}n(n-3)}} \int d\delta_{ij} \prod_{1 \leq i < j \leq n} \Gamma(\delta_{ij}) (P_{ij})^{-\delta_{ij}} \int_{-i\infty}^{+i\infty} \prod_{i=1}^n \frac{dc_i}{(2\pi i)^n} \\ \times \left(\left(\prod_{i=1}^n \Gamma\left(\frac{\Delta_{B_i} + (h + c_i) - 2h}{2}\right) \right) \Gamma\left(\frac{\sum_{i=1}^n (h - c_i) - 2h}{2}\right) f_{\Delta_i}(c_i) \mathcal{M}(k_i, c_i) \right), \quad (3.18)$$

where we introduce a new notation $\mathcal{M}(k_i, c_i)$ and call it as the Mellin integrand which is given as

$$\mathcal{M}(k_i, c_i) = \int \prod_{i=1}^n \frac{ds_i}{s_i} \frac{d\bar{s}_i}{\bar{s}_i} s_i^{h+c_i+a_i} \bar{s}_i^{h-c_i+a_i} (g_i)^{b_i} (1 + s_i^2 F_i)^{d_i}, \quad (3.19)$$

such that the Mellin amplitude can be given in terms of the Mellin integrand as,

$$M(\delta_{ij}) = \int_{-i\infty}^{+i\infty} \prod_{i=1}^n \frac{dc_i}{(2\pi i)^n} \left(\prod_{i=1}^n \Gamma\left(\frac{\Delta_{B_i} + (h + c_i) - 2h}{2}\right) \right) \\ \times \Gamma\left(\frac{\sum_{i=1}^n (h - c_i) - 2h}{2}\right) f_{\Delta_i}(c_i) \mathcal{M}(k_i, c_i). \quad (3.20)$$

Note that in the Mellin integrand $\mathcal{M}(k_i, c_i)$ we have used k_i instead of δ_{ij} , and recall that $\delta_{ij} \equiv k_i \cdot k_j$.

Furthermore, a few words about the exponents a_i, b_i and d_i are in order. With respect to the i_{th} propagator in Fig. 2, a_i is the product of the momenta flowing through the propagator i from both sides. So according to our convention of Fig. 2, where K_i is the sum of all momenta of the fields contained in the block B_i , i.e. $K_i \equiv k_{a_i} + \dots + k_{a_i+n-1}$, we get

$$a_i \equiv -(K_i \cdot \sum_{m \neq i} K_m) = K_i^2. \quad (3.21)$$

⁶ In the embedding formalism, $P_{ab} = -2P_a \cdot P_b$ and $P_a^2 = 0$

The exponent b_i is the sum of all possible products of the momenta flowing through the propagators connecting the propagator i on the \bar{s}_i side i.e.

$$b_i \equiv -\left(\sum_{m,n \neq i, m \neq n} K_m \cdot K_n\right), \quad (3.22)$$

while d_i is the sum of all possible products of the momenta flowing from the other direction, namely s_i side,

$$d_i \equiv -\frac{a_i - \Delta_{B_i}}{2}, \quad (3.23)$$

recall that $\Delta_{B_i} = \sum_{j \in B_i} \Delta_j$.

Integrals over s_i and \bar{s}_i

The integral $\mathcal{M}(k_i, c_i)$ in (3.19) can be greatly simplified using a set of transformations which are the generalization of the transformations used in [24]. Firstly, we rescale s_i^2 by a factor of F_i , then (3.19) becomes

$$\mathcal{M}(k_i, c_i) = \prod_{i=1}^n \int \frac{ds_i}{s_i} s_i^{h+c_i+a_i} (1+s_i^2)^{d_i} \int \frac{d\bar{s}_i}{\bar{s}_i} F_i^{-\frac{h+c_i+a_i}{2}} \bar{s}_i^{h-c_i+a_i} g_i^{b_i}. \quad (3.24)$$

Then we can make a set of consecutive transformations on \bar{s} 's, to simplify the integral further,

$$\begin{aligned} \bar{s}_j^2 &\rightarrow x_j \\ x_2 &\rightarrow \frac{x_2}{1+x_1} \\ &\vdots \\ x_n &\rightarrow \frac{x_n}{(1+x_1)(1+x_2)\dots(1+x_{n-1})} \\ x_1 &\rightarrow \frac{x_1}{(1+x_2)(1+x_3)\dots(1+x_n)} \\ &\vdots \\ x_{n-1} &\rightarrow \frac{x_{n-1}}{1+x_n}. \end{aligned} \quad (3.25)$$

Under the above set of transformations we find that g_i , F_i and x_i (or \bar{s}_i^2) transform as,

$$\begin{aligned} g_i &\rightarrow \frac{1 + \sum_{j=i}^n x_j}{1 + \sum_{j=i+1}^n x_j}, \\ F_i &\rightarrow \frac{(1+x_i)(1 + \sum_{j=i+1}^n x_j)}{1 + \sum_{j=i}^n x_j}, \\ x_i &\rightarrow \frac{x_i(1 + \sum_{j=i}^n x_j)}{(1 + \sum_{j=i+1}^n x_j)(1 + \sum_{j=1}^n x_j)}. \end{aligned} \quad (3.26)$$

We also find that the exponent of $(1 + \sum_{j=i+1}^n x_j)$ is given by

$$\left(\frac{h + c_i + a_i + h - c_i + a_i}{2} + b_i\right) - \left(\frac{h + c_{i-1} + a_{i-1} + h - c_{i-1} + a_{i-1}}{2} + b_{i-1}\right),$$

and this vanishes when we use the definition of a_i and b_i from (3.21) and (3.22). Hence all the terms of the form $(1 + \sum_{j=i+1}^n x_j)$ do not have any contribution to the exponent. Finally we are left with a very simple integral given as

$$\mathcal{M}(k_i, c_i) = \prod_{i=1}^n \int \frac{ds_i}{s_i} s_i^{h+c_i+a_i} (1 + s_i^2)^{d_i} \int \frac{dx_i}{x_i} x_i^{\frac{h-c_i+a_i}{2}} (1 + x_i)^{-\frac{h+c_i+a_i}{2}} (1 + \sum_{j=1}^n x_j)^q, \quad (3.27)$$

where $q = \frac{1}{2}(\sum c_i - (n-2)h)$.

The s_i integrals give the Gamma functions,

$$\prod_{i=1}^n \int \frac{ds_i}{s_i} s_i^{h+c_i+a_i} (1 + s_i^2)^{d_i} = \prod_{i=1}^n \frac{\Gamma(\frac{h+c_i+a_i}{2}) \Gamma(\frac{\Delta_{B_i}-c_i-h}{2})}{\Gamma(\frac{\Delta_{B_i}-a_i}{2})}, \quad (3.28)$$

where we have used the fact that $k_i^2 = -\Delta_i$ and also the definition of a_i and d_i from (3.21) and (3.23) to get the final form of the result.

To perform the x_i integrals, we will do a series expansion of the factor $(1 + \sum_{j=1}^n x_j)^q$ in (3.27) as,

$$\begin{aligned} (1 + \sum_{j=1}^n x_j)^q &= \sum_{m_1, \dots, m_n=0}^{\infty} \prod_{k=1}^n (-q + \sum_{j=1}^{k-1} m_j)_{m_k} \prod_{k=1}^n \frac{(-x_k)^{m_k}}{m_k!} \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} \prod_{k=1}^n (-q)_{\sum_i m_i} \prod_{k=1}^n \frac{(-x_k)^{m_k}}{m_k!}. \end{aligned} \quad (3.29)$$

Then the x_i integrals can be performed easily, which leads to

$$\begin{aligned} &\prod_{i=1}^n \int \frac{dx_i}{x_i} x_i^{\frac{h-c_i+a_i}{2}} (1 + x_i)^{-\frac{h+c_i+a_i}{2}} (1 + \sum_{j=1}^n x_j)^q \\ &= F_A^{(n)} \left(-q, \left\{ \frac{h-c_1+a_1}{2}, \dots, \frac{h-c_n+a_n}{2} \right\}, \{1-c_1, \dots, 1-c_n\}; 1, \dots, 1 \right) \\ &\times \prod_{i=1}^n \frac{\Gamma(c_i) \Gamma(\frac{h-c_i+a_i}{2})}{\Gamma(\frac{h+c_i+a_i}{2})}. \end{aligned} \quad (3.30)$$

So the Mellin integrand (3.27) is now given by the product of (3.28) and (3.30).

We can now do the final integration over the c variables to get the Mellin amplitude,

$$M(k_i) = \int_{-i\infty}^{+i\infty} \left(\prod_{i=1}^n \frac{dc_i}{2\pi i} f_{\Delta_i}(c_i) \Gamma\left(\frac{\Delta_{B_i} + c_i - h}{2}\right) \Gamma\left(\frac{(n-2)h - \sum_{i=1}^n c_i}{2}\right) \mathcal{M}(k_i, c_i) \right). \quad (3.31)$$

As pointed out in [24], we can do this integral by determining the poles in the kinematics, namely, the a_i 's and their corresponding residues. They can be determined by pinching of the contour by

two poles, $c_i = \pm(\Delta_i - h)$ from $f_{\Delta_i}(c_i)$ and $c_i = a_i + h + 2n_i$ from $\Gamma(\frac{h-c_i+a_i}{2})$ with positive integer n_i , for each c_i integration. The above mentioned residues can be cast in a simple form and we can write the full result for (3.31) in the following form,

$$M(k_i) = \sum_{n_1, \dots, n_n=0}^{\infty} \left(\prod_{i=1}^n \mathcal{P}_i \right) V_{[0, \dots, 0, n_1]}^{\Delta_1, \dots, \Delta_{n-1}, \Delta_{n_1}} \dots V_{[0, \dots, 0, n_n]}^{\Delta_{(n-1)^2+1}, \dots, \Delta_{n(n-1)}, \Delta_{n_n}} V_{[n_1, \dots, n_n]}^{\Delta_{n_1}, \dots, \Delta_{n_n}}, \quad (3.32)$$

where the simple poles in a_i can be read off from the terms, $\frac{1}{a_i + (\Delta_{n_i} + 2n_i)}$, appearing in \mathcal{P}_i .

One may worry about other possible poles, including the poles $\pm c_i = \Delta_{B_i} - h + 2m$ from $\Gamma(\frac{\Delta_{B_i} \pm c_i - h}{2})$, and the pole from $\Gamma(\frac{(n-2)h - \sum_i c_i}{2})$. Firstly the pole from $\Gamma(\frac{(n-2)h - \sum_i c_i}{2})$ is canceled by $(-q)_{\sum n_i}$ in the Lauricella function, and as for the other pole, we note that after pinching off $c_i = -(\Delta_{B_i} - h + 2m)$ with $c_i = a_i + h + 2n_i$, this pole is canceled out by $\Gamma(\frac{\Delta_{B_i} + a_i}{2})$ in $I_s(c_i)$.

Furthermore, it has been argued in [25] that the correlation function in Mellin space has good behavior at large a_i , and poles and the corresponding residues are enough to determine the whole function, so (3.32) is the complete result of the integral (3.31), and it leads to the Feynman rules stated earlier in section 2.

For a ϕ^n theory, there are also vertices with less than n off-shell legs. In fact one can have vertices with $n, (n-1), \dots, 1$ and 0 off-shell legs. We can obtain the results of these cases from the vertex with a maximal number of off-shell legs in Fig. 2 by taking some of B_i 's to be a single leg connecting directly to the boundary. The result of this Witten diagram can be obtained by simply removing s_n and \bar{s}_n and noticing that for a single leg on the boundary we have $(t_n P_n)^2 = 0$. If we take m out of n B_i 's to be single legs, the result is actually in the same form of the ϕ^{n-m} theory, as one would have expected.

C. General case

Let us consider the most general case of a maximal off-shell vertex in an arbitrary Witten diagram in a scalar ϕ^n theory, as in Fig. 3. Here, the off-shell leg Q_k , is connected to the set of on-shell fields in the block named Z_k with $k = 1, \dots, n$ via many propagators and vertices. All these intermediate propagators and vertices are collectively denoted by the blob attached to Q_k . We also assume that we had already done the Q integrations for all the propagators inside each blob connected to the block Z_k and the contribution from this to the exponent in the integrand of the s, \bar{s} , and c integrals is labeled as \mathcal{B}_k^Z . We note that this quantity depends on the variables s, \bar{s} associated with all the propagators in the blob of the k_{th} block Z_k and all the $t_k P_k$'s associated with

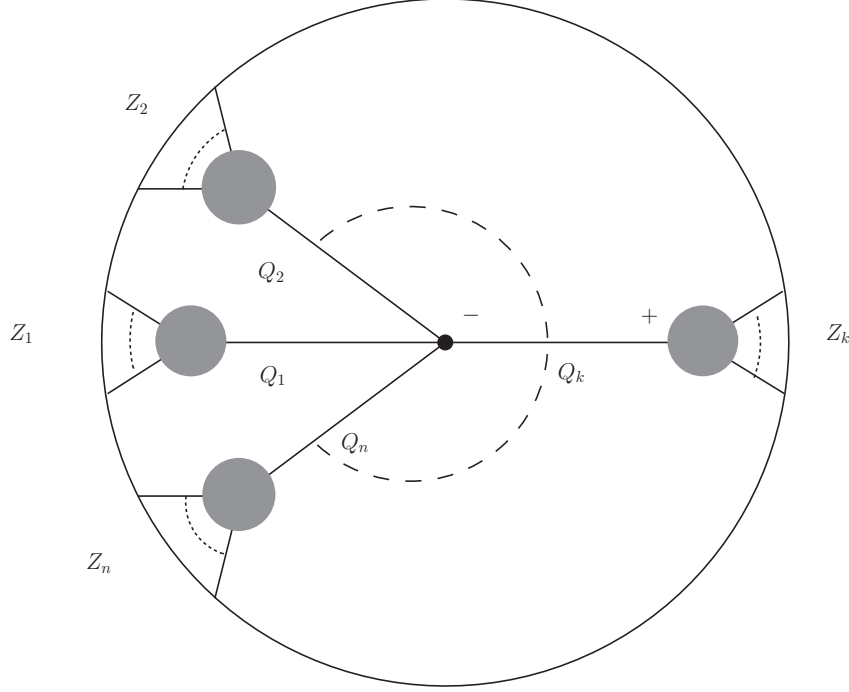


FIG. 3: A vertex with all off-shell legs in arbitrary Witten diagram.

the on-shell fields contained in this block,⁷ but most importantly it does not contain the variables associated with the propagator Q_k i.e. s_k and \bar{s}_k .

We note that Fig. 2 is a special case of Fig. 3 if the k_{th} blob contains only one maximal vertex of ϕ^n theory with $(n-1)$ on-shell legs and then $\mathcal{B}_k^Z \equiv B_k$.

Now it is obvious that just as in the previous section, the contribution to the Mellin exponent after doing the integrations over Q_1 to Q_n can be written as,

$$E_{Q_n} = \sum_{i'} (\mathcal{D}_{i'})^2 + \sum_{l=1}^n (\bar{s}_l \mathcal{Y}_{l-1}^Z + s_l \mathcal{B}_l^Z)^2, \quad (3.33)$$

where

$$\mathcal{Y}_n^Z = \sum_{i=1}^n \frac{(\prod_{k=1}^n g_k)}{g_i} s_i \bar{s}_i \mathcal{B}_i^Z \quad (3.34)$$

and $\mathcal{D}_{i'}$ is defined as follows. Since the Q_1 to Q_n integrations affect the form of \mathcal{B}_i^Z , obtained from the previous Q integrations, at each step of these integrations we will get some complicated functions which we denote as $\mathcal{D}_{i'}$. We note that these do not have any dependence on the s and \bar{s} variables associated with the maximal vertex and are not of any interest for the remaining

⁷ We will label all the variables associated with the propagators in the blob with a primed index.

calculation⁸. The other definitions being the same as in (3.14). Even though, it seems that this situation is much more complicated than before we note that none of the \mathcal{B}_k^Z 's depend on the s_k and \bar{s}_k 's associated with the vertex under consideration and moreover the \mathcal{B}_k^Z 's are also linear in the $t_k P_k \in Z_k$ and hence we can apply Symanzik star formula, to perform the t_i integrals, as before by expanding the square. Now, let us focus on the second sum in (3.33) and since it has the same form as (3.15), the analysis is similar to the one in the previous section but with a few added subtleties. In particular, when applying Symanzik star formula, the contributions from the $\mathcal{B}_i^Z \cdot \mathcal{B}_j^Z$ term will be same as that of the $B_i \cdot B_j$ term before, with $i \neq j$, but the contributions from $\mathcal{B}_i^Z \cdot \mathcal{B}_i^Z$ would be different from the analogous term in the previous section.⁹

So all of the analysis in the previous section still holds as we can isolate the integrations for this particular vertex and write the Mellin integrand as,

$$\mathcal{M}(k_j) = \int ds \mathcal{V}(s) \int \prod_{j=1}^n \frac{ds_j}{s_j} \frac{d\bar{s}_j}{\bar{s}_j} s_j^{h+c_j+a_j} \bar{s}_j^{h-c_j+a_j} g_j^{-b_j} \prod_{j=1}^n H_j(s_j^2 F_j, s), \quad (3.35)$$

where we denote $\int ds \mathcal{V}(s)$ as the integrals irrelevant to the vertex. Even though $H_i(s_i^2 F_i, s)$ can be a complicated function, for the s and \bar{s} relevant to the vertex we are interested in, they are always of the form $s_j^2 F_j$. So we can rescale s_i^2 by a factor of F_i for $i = 1, 2, \dots, n$,¹⁰ and after rescaling, $H_i(s_i^2 F_i, s)$ will be included in the irrelevant integral $\int ds \mathcal{V}(s)$. So at the end we arrive at the integral related to the vertex we are interested in i.e. the \bar{s} dependent part

$$\mathcal{M}(k_i)|_{\bar{s}} = \int \prod_{i=1}^n \frac{d\bar{s}_i}{\bar{s}_i} \bar{s}_i^{h-c_i+a_i} g_i^{-b_i} \prod_{i=1}^n F_i^{\frac{-h+c_i+a_i}{2}}, \quad (3.36)$$

which is exactly the same as the \bar{s} part of the integral in (3.24) and hence gives the same form of the maximal off-shell vertex.

IV. FLAT SPACE LIMIT

In this section we consider the flat space limit of the Mellin space Feynman rules. We will show that these rules give rise to the usual Feynman rules for scattering amplitudes in the flat space limit. This limit can also be considered as a consistency check of the AdS Feynman rules. The flat

⁸ In Appendix. B we do a specific example of a general Witten diagram and there we also give an explicit form of these $\mathcal{D}_{i'}$'s for that special case.

⁹ The reason is that some terms in $\mathcal{B}_i^Z \cdot \mathcal{B}_i^Z$ could mix the contributions from the first sum, $\sum_{i'} (\mathcal{D}_{i'})^2$, in (3.33), however there is no such kind of mixing for the term of the form $\mathcal{B}_i^Z \cdot \mathcal{B}_j^Z$, because there cannot be any term in $\sum_{i'} (\mathcal{D}_{i'})^2$ to have this form, since the i th blob had never talked to j th blob before.

¹⁰ Note $F_n = 1$, so when $j = n$ there is no rescaling.

space limit corresponds to the large δ_{ij} behavior of the Mellin amplitudes. As had been discussed in [23] and [25], in this limit the Mellin amplitudes are related to the S-matrix in flat space by the following relation¹¹

$$M(\delta_{ij}) \approx \int_0^\infty d\beta \beta^{\frac{1}{2} \sum \Delta_i - h - 1} e^{-\beta} T(p_i \cdot p_j = 2\beta\delta_{ij}) , \quad \delta_{ij} \gg 1 , \quad (4.1)$$

where $T(p_i \cdot p_j = 2\beta\delta_{ij})$ is the flat space S-matrix as a function of the kinematic invariants $p_i \cdot p_j$ and β is an integration parameter. We will study the case of large δ_{ij} limit with Δ_i fixed. In order to confirm that the AdS Feynman rules indeed reduce to the usual flat space Feynman rules of ϕ^n theory in this limit, we only need to show that

$$\sum_{\{n_i\}} M(n_1, \dots, n_s) = \frac{(-1)^s}{2^s} \Gamma\left(\frac{1}{2} \sum \Delta_i - h - s\right), \quad (4.2)$$

where $\sum_{\{n_i\}} M(n_1, \dots, n_s)$ related to the Mellin amplitude by

$$M(\delta_{ij}) \approx \sum_{\{n_i\}} M(n_1, \dots, n_s) \prod_{i=1}^s \frac{1}{k_i^2}, \quad (4.3)$$

where $\frac{1}{k_i^2}$ is the propagator, s is the number of propagators, and the summation over $\{n_i\}$ become clear shortly. The above equation (4.2) follows directly from (4.1) by using the definition of the flat space massless scalar scattering amplitudes.

We will use the AdS Feynman rules (2.3) to compute the left hand side of (4.1). As in the case of ϕ^3 theory [25], we can always start from the bulk propagators closer to the external legs in the Witten diagram and perform the sum over $\{n_i\}$ recursively. To do so for a general ϕ^n theory, we need the following identity, which will be proved shortly,

$$\begin{aligned} & \sum_{n_{I_1}, n_{I_2}, \dots, n_{I_m}=0}^\infty \frac{V^{(1)}(n_{I_1}) V^{(1)}(n_{I_2}) \dots V^{(1)}(n_{I_m}) V^{(m+1)}(n_{I_1}, n_{I_2}, \dots, n_{I_m}, n_o)}{P_{n_{I_1}} P_{n_{I_2}} \dots P_{n_{I_m}}} \\ &= \frac{(-1)^m}{2^m} \left((m+1) - \frac{\sum \Delta_i}{2} + \Delta_{n_o} \right) \Gamma\left(\frac{\sum \Delta_i}{2} - h - m\right), \end{aligned} \quad (4.4)$$

where $P_n \equiv -2n! \Gamma(1 + \Delta_n + n - h)$, and $V^{(1)}(n_I)$ denotes the vertex with one off-shell leg where this leg is labeled as n_I , and we follow a similar logic to define $V^{(m+1)}(n_{I_1}, n_{I_2}, \dots, n_{I_m}, n_o)$, which denotes the vertex with $(m+1)$ off-shell legs. Finally, the summation in $\sum \Delta_i$ indicates the sum over all the external on-shell legs. The identity (4.4) can be proved by performing the summation in the following order: first we sum over l_1 , the summation variable in the Lauricella function $F_A^{(n)}$

¹¹ Notice it is slightly different from Eq. (127) in [25], because we define the Mellin amplitudes by a different normalization factor.

then the corresponding n_{I_1} , next we do the sum over l_2 then n_{I_2} and so on. At each step of the sum we can apply the identity,

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} = \frac{(c-b)_{-a}}{(c)_{-a}}. \quad (4.5)$$

We note here that $((m+1) - \frac{\sum \Delta_i}{2} + \Delta_{n_o})_{n_o}$ in (4.4) has the same dependence on n_o as $V^{(1)}(n)$ does. So the summation on n_o in a general Witten diagram can be done by applying the above identities again. At the end of the day, we will be left with a sum involving only factors of the form of $V^{(1)}(n)$. It can be shown that the answer for the final sum of any Witten diagram is indeed given as Eq. (4.2).

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Appendix A: Useful integrals

Here we list some formulas, which have been extensively used in this paper. For more details about these formulas, see [23, 39].

X integral formula

$$\int_0^{+\infty} \prod_i \left(\frac{dt_i}{t_i} t_i^{\alpha_i} \right) \int_{AdS} dX e^{2T \cdot X} = \pi^h \Gamma \left(\frac{\sum_i \alpha_i - 2h}{2} \right) \int_0^{+\infty} \prod_i \left(\frac{dt_i}{t_i} t_i^{\alpha_i} \right) e^{T^2}, \quad (A1)$$

where $T = \sum_i t_i P_i$.

Q_i integral formula

$$\int_0^{+\infty} \frac{ds}{s} \frac{d\bar{s}}{\bar{s}} s^{h+c} \bar{s}^{h-c} \int_{\partial AdS} dQ e^{2T \cdot Q} = 2\pi^h \int_0^{+\infty} \frac{ds}{s} \frac{d\bar{s}}{\bar{s}} s^{h+c} \bar{s}^{h-c} e^{T^2}, \quad (\text{A2})$$

where $T = (sX + \bar{s}Y)$.

t integral formula

This is also called the Symanzik star integration formula where we consider a set of n points in Euclidean space x_i and their differences $x_i - x_j$. In the embedding formalism we have $P_{ij} \equiv -2P_i \cdot P_j = (x_i - x_j)^2$. Then Symanzik's formula is:

$$\int_0^{+\infty} \left(\prod_{i=1}^n \frac{dt_i}{t_i} t^{\Delta_i} \right) e^{-\left(\sum_{1 \leq i < j \leq n} t_i t_j P_{ij} \right)} = \frac{\pi^h/2}{(2\pi i)^{\frac{1}{2}n(n-3)}} \int d\delta_{ij} \prod_{1 \leq i < j \leq n} \Gamma(\delta_{ij}) (P_{ij})^{-\delta_{ij}} \quad (\text{A3})$$

where the integration is over $n(n-3)/2$ variables and the integration paths are chosen parallel to the imaginary axis, with real parts such that the real parts of the arguments of the gamma functions are positive.

Appendix B: Example: 9-points in ϕ^3 theory

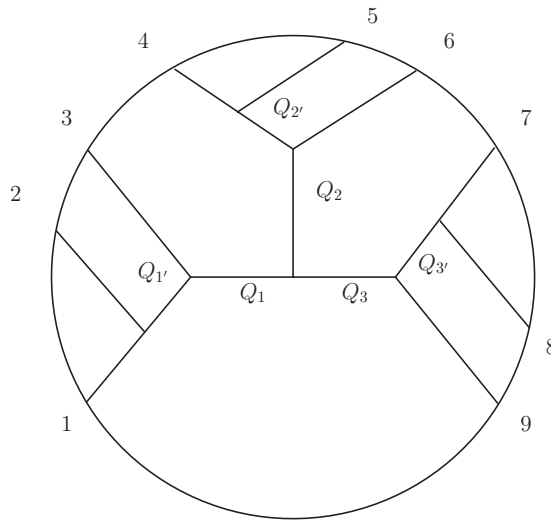


FIG. 4: A vertex with all off-shell legs in 9-point amplitude in ϕ^3 theory

In order to illustrate the general strategy of isolating the maximal vertex in any arbitrary Witten diagram, as discussed in section 3.3, here we will study a specific example, that of a 9-point Witten diagram in ϕ^3 theory and we will write down the results as a special case of (3.35) and (3.36). We have the Q variables labeled as $Q_{1'}, Q_{2'}, Q_{3'}, Q_1, Q_2, Q_3$ and then we will integrate them out in that particular order. We note here that only the variables Q_1, Q_2, Q_3 are relevant for the vertex under consideration. After doing all the Q integrals we will get the exponent as before given by (3.33),

$$E_{Q_3} = \sum_{i=1}^3 (\mathcal{D}_{i'})^2 + \sum_{l=1}^3 (\bar{s}_l \mathcal{Y}_{l-1}^Z + s_l \mathcal{B}_l^Z)^2, \quad (\text{B1})$$

where the term in the first bracket is given by

$$\begin{aligned} \mathcal{D}_{1'} &= P_3 t_3 \bar{s}_{1'} + s_{1'} (P_1 t_1 + P_2 t_2), \\ \mathcal{D}_{2'} &= P_6 t_6 \bar{s}_{2'} + s_{2'} (P_4 t_4 + P_5 t_5), \\ \mathcal{D}_{3'} &= P_9 t_9 \bar{s}_{3'} + s_{3'} (P_7 t_7 + P_8 t_8), \end{aligned} \quad (\text{B2})$$

and these terms do not contain any of the variables relevant to the vertex we are interested in and hence we will not consider them further. Now let us focus on the term inside the second bracket where the relevant functions \mathcal{B}^Z and \mathcal{Y}^Z , defined by (3.34), are given by,

$$\begin{aligned} \mathcal{B}_1^Z &= \mathcal{D}_{1'} \bar{s}_{1'} + P_3 t_3, \\ \mathcal{B}_2^Z &= \mathcal{D}_{2'} \bar{s}_{2'} + P_6 t_6, \\ \mathcal{B}_3^Z &= \mathcal{D}_{3'} \bar{s}_{3'} + P_9 t_9 \end{aligned} \quad (\text{B3})$$

and

$$\begin{aligned} \mathcal{Y}_0^Z &= 0, \\ \mathcal{Y}_1^Z &= s_1 \mathcal{B}_1^Z \bar{s}_1, \\ \mathcal{Y}_2^Z &= s_2 \mathcal{B}_2^Z (\bar{s}_1^2 + 1) \bar{s}_2 + s_1 \mathcal{B}_1^Z \bar{s}_1 ((\bar{s}_1^2 + 1) \bar{s}_2^2 + 1). \end{aligned} \quad (\text{B4})$$

The next step is to expand the second term of E_{Q_3} like before and after doing all the t_i integrals we get the form of the integrand as in (3.35),

$$\mathcal{M}(k_j) = \int ds \mathcal{V}(s) \int \prod_{j=1}^3 \frac{ds_j}{s_j} \frac{d\bar{s}_j}{\bar{s}_j} s_j^{h+c_j+a_j} \bar{s}_j^{h-c_j+a_j} g_j^{-b_j} \prod_{j=1}^3 H_j(s_j^2 F_j, s), \quad (\text{B5})$$

where g and F are defined by (3.14) and (3.17) and a_j, b_j are defined by (3.21) and (3.22) respectively. Moreover, the part which is irrelevant to the maximal vertex is given by

$$\begin{aligned} \mathcal{V}(s) = & \prod_{j=1'}^{3'} \frac{ds_j}{s_j} s_j^{h+c_j+a'_j} \bar{s}_j^{h-c_j+a'_j} (\bar{s}_{1'}^2 + 1)^{-k_3(k_1+k_2+k_3)} \\ & \times (\bar{s}_{2'}^2 + 1)^{-k_6(k_4+k_5+k_6)} (\bar{s}_{3'}^2 + 1)^{-k_9(k_7+k_8+k_9)}, \end{aligned} \quad (\text{B6})$$

where,

$$\begin{aligned} a'_{1'} &= -(k_1 + k_2)^2, \\ a'_{2'} &= -(k_4 + k_5)^2, \\ a'_{3'} &= -(k_7 + k_8)^2. \end{aligned} \quad (\text{B7})$$

The complicated H function which would eventually give the terms relevant for the vertex, is given by,

$$\begin{aligned} H_1(s_1^2 F_1, s) &= (F_1 s_1^2 (\bar{s}_{1'}^2 + 1) + 1)^{(k_1+k_2)k_3} (s_{1'}^2 (F_1 s_1^2 \bar{s}_{1'}^2 + 1) + 1)^{k_1 k_2}, \\ H_2(s_2^2 F_2, s) &= (F_2 s_2^2 (\bar{s}_{2'}^2 + 1) + 1)^{(k_4+k_5)k_6} (s_{2'}^2 (F_2 s_2^2 \bar{s}_{2'}^2 + 1) + 1)^{k_4 k_5}, \\ H_3(s_3^2 F_3, s) &= (F_3 s_3^2 (\bar{s}_{3'}^2 + 1) + 1)^{(k_7+k_8)k_9} (s_{3'}^2 (F_3 s_3^2 \bar{s}_{3'}^2 + 1) + 1)^{k_7 k_8}. \end{aligned} \quad (\text{B8})$$

Now, as before, we can use the transformation

$$s_i^2 \rightarrow \frac{\bar{s}_i^2}{F_i} \quad (\text{B9})$$

and we will get the required form of the \bar{s} integral part as in (3.36), to be

$$\mathcal{M}(k_i)|_{\bar{s}} = \int \prod_{i=1}^3 \frac{d\bar{s}_i}{\bar{s}_i} \bar{s}_i^{h-c_i+a_i} g_i^{-b_i} F_i^{\frac{-h+c_i+a_i}{2}}. \quad (\text{B10})$$

From (B8) we also see that after the rescaling of s_i^2 's we are left with a term of the form ,

$$\begin{aligned} \mathcal{V}'(s) = & (s_1^2 (\bar{s}_{1'}^2 + 1) + 1)^{(k_1+k_2)k_3} (s_{1'}^2 (s_1^2 \bar{s}_{1'}^2 + 1) + 1)^{k_1 k_2} \\ & \times (s_2^2 (\bar{s}_{2'}^2 + 1) + 1)^{(k_4+k_5)k_6} (s_{2'}^2 (s_2^2 \bar{s}_{2'}^2 + 1) + 1)^{k_4 k_5} \\ & \times (s_3^2 (\bar{s}_{3'}^2 + 1) + 1)^{(k_7+k_8)k_9} (s_{3'}^2 (s_3^2 \bar{s}_{3'}^2 + 1) + 1)^{k_7 k_8}, \end{aligned} \quad (\text{B11})$$

which we note is independent of the \bar{s} variables related to the maximal vertex and hence irrelevant.

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